Government of **Western Australia**
Curriculum Council

WACE
Western Australian Certificate of Education

MATHEMATICS: SPECIALIST

3C/3D
Calculator-free

WACE Examination 2011

Marking Key

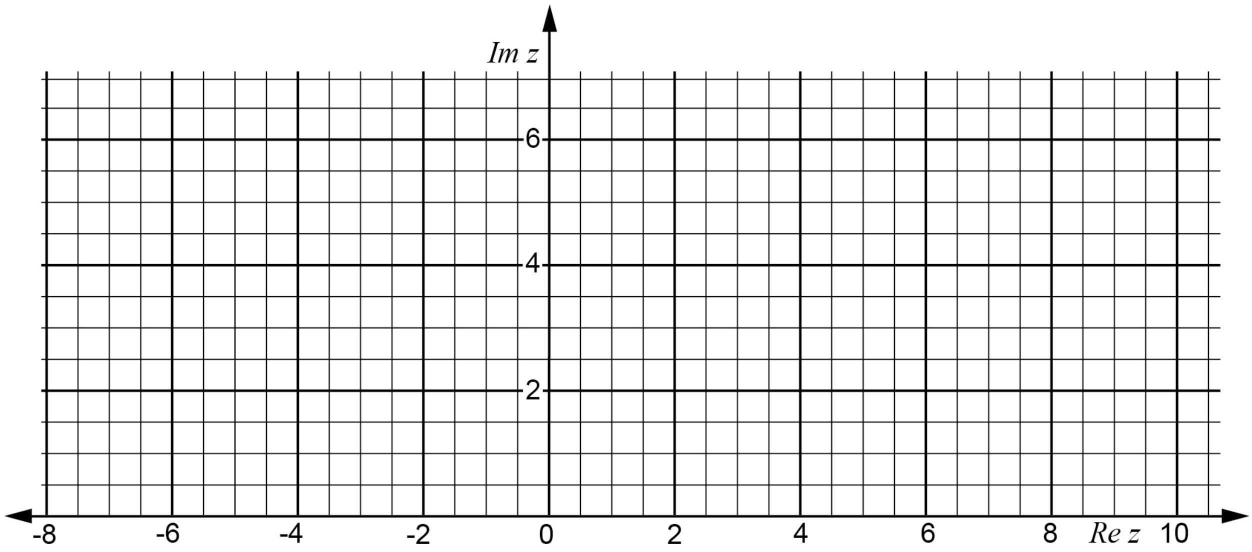
Marking keys are an explicit statement about what the examiner expects of candidates when they respond to a question. They are essential to fair assessment because their proper construction underpins reliability and validity.

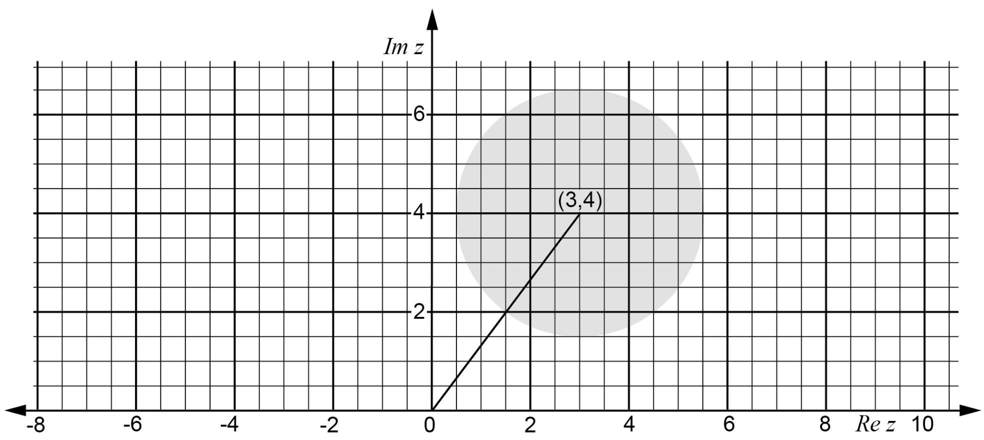
When examiners design an examination, they develop provisional marking keys that can be reviewed at a marking key ratification meeting and modified as necessary in the light of candidate responses.

Section One: Calculator-free **40 Marks**

Question 1 **(5 marks)**

(a) Sketch, on the complex plane below, the region defined by $|z-3-4i| \le \frac{5}{2}$. (3 marks)



Solution

Specific behaviours
✓ correctly shades a circular disk ✓ gives correct coordinates for the centre of the circle and ✓ circumference passes through at least three of: (3, 1.5), (5.5,4), (3, 6.5) and (0.5, 5)

(b) For the region in (a), find the maximum value of $|z|$. (2 marks)

Solution
Maximum value of $\lvert z\rvert$ = the distance from the origin to the centre + the length of the radius i.e. Maximum value of $\lvert z\rvert = 5 + \frac{5}{2} = \frac{15}{2}$
Specific behaviours
✓ calculates the distance from the origin to the centre of the circle ✓ states the correct answer Note ✓ shows understanding of process but wrong answer

Question 2 **(3 marks)**

Use proof by exhaustion to prove that no square number ends in 8. (3 marks)

Solution

x	0	1	2	3	4	5	6	7	8	9
x^2	0	1	4	9	16	25	36	49	64	81

No number from 0 to 9 inclusive has a square that ends in 8. Any number greater than 9 has a units digit equal to one of those shown in the first line of the table. When squared, the end digit will be the same as the corresponding end digit in the second line of the table. Thus no square number ends in 8.

Specific behaviours

✓ demonstrates that no number from 0 to 9 inclusive has a square which ends in 8

✓ $(x+10)^2$ indicates the place value of each group

✓ the units digit is always filled by the x^2 unit

Or

✓ calculates squares from 0 to 9 or 1 to 10

✓ calculates squares from 10 to 19 (or 11 to 20) and compares end-digits with above

✓ correctly explains why the pattern continues

Question 3 **(6 marks)**

Determine the following integrals:

(a) $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta$ (3 marks)

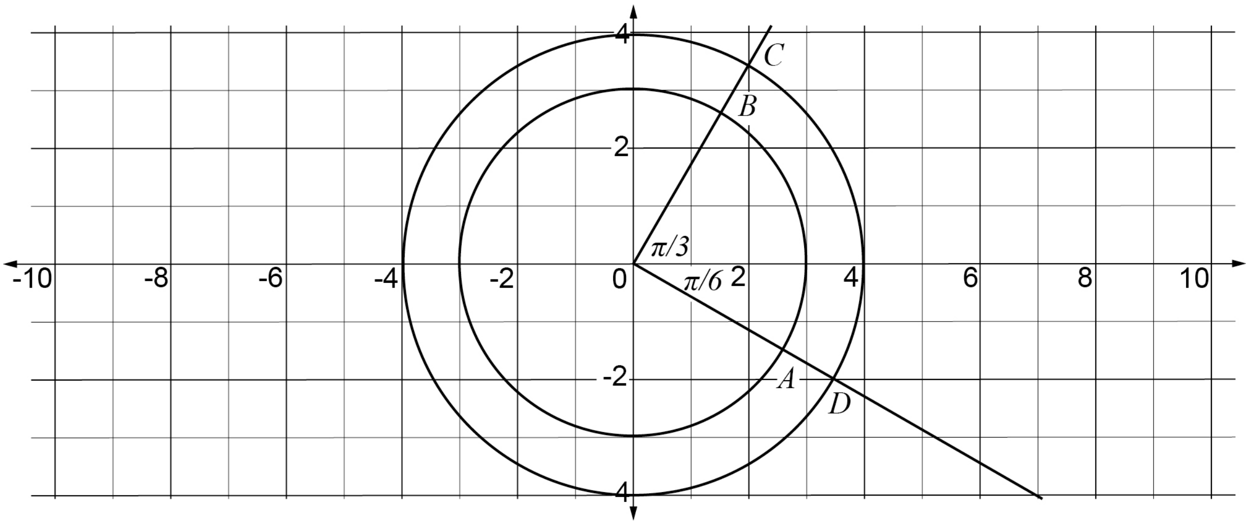
Solution
$\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta=\frac{1}{2}\int\frac{\sin\theta}{\cos\theta+1}\,d\theta$ Hence $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta=-\frac{1}{2}\ln\lvert\cos\theta+1\rvert+c$ Or $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta=\frac{1}{2}\int\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$ i.e. $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta=\int\frac{1}{u}\,du$ where $u=\sin\frac{\theta}{2}$ Hence $\displaystyle\int \frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta+1}\,d\theta=\ln\lvert u\rvert+c=\ln\left\lvert\sin\frac{\theta}{2}\right\rvert+c$
Specific behaviours
✓ replaces $\sin\frac{\theta}{2}\cos\frac{\theta}{2}=\frac{1}{2}\sin\theta$ ✓ gives logarithmic solution ✓ provides correct coefficient $\left(-\frac{1}{2}\right)$, including c Or ✓ replaces $\cos\theta+1=\sin^2\frac{\theta}{2}$ ✓ gives logarithmic solution ✓ provides correct solution, including c

(b) $\int \cos^3 x\,dx$ (3 marks)

Solution
$\int \cos^3 x\,dx = \int \left(1-\sin^2 x\right)\ \cos x\,dx$ i.e. $\int \cos^3 x\,dx = \int \cos x - \sin^2 x \cos x\,dx$ Let $u = \sin x \Rightarrow \int \cos x\,dx - \int u^2 du$ Hence $\int \cos^3 x\,dx = \sin x - \dfrac{\sin^3 x}{3} + c$ Or $\int \cos^3 x\,dx = \dfrac{1}{4}\int \left(\cos 3x + 3\cos x\right)dx$ Hence $\int \cos^3 x\,dx = \dfrac{1}{4}\left(\dfrac{\sin 3x}{3} + 3\sin x\right) + c$
Specific behaviours
✓ recognises that $\cos^3 x$ may replaced by $\left(1-\sin^2 x\right)\ \cos x$ ✓ uses the substitution $u = \sin x$ or by inspection ✓ integrates correctly Or ✓ recognises that $\cos^3 x$ may be replaced by $\frac{1}{4}\left(\cos 3x + 3\cos x\right)$ ✓✓ Integrates correctly for each term

Question 4 **(6 marks)**

(a) Use polar inequalities to describe the region bounded by the minor arcs AB and CD and the straight lines BC and AD in the diagram below. (2 marks)



Solution
$3 \le r \le 4$ and $-\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{3}$
Specific behaviours
✓ identifies inequalities with both radii using polar notations ✓ identifies inequalities with both angles using polar notations * Note ✓ for correct radii and angles using other notation

(b) If the graph of $r = k\theta,\ k > 0$, passes through A, find a possible value for k. (2 marks)

Solution
A may be described as the point with polar coordinates $\left(3,\ \dfrac{11\pi}{6}\right)$. For this value of θ, $3 = \dfrac{11k\pi}{6}$ i.e. $k = \dfrac{18}{11\pi}$
Specific behaviours
✓ identifies possible coordinates for A ✓ solves for k

(c) Find the distance between B and D. (2 marks)

Solution
$\left\|\overrightarrow{OB}\right\| = 3$; $\left\|\overrightarrow{OD}\right\| = 4$ and $\angle BOD = \frac{\pi}{2}$ Hence $\left\|\overrightarrow{BD}\right\| = 5$ units
Specific behaviours
✓ identifies the measurements of triangle OAB ✓ solves for $\left\|\overrightarrow{BD}\right\|$

Or

Solution
B has Cartesian coordinates $\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right)$; D has Cartesian coordinates $\left(2\sqrt{3}, -2\right)$ Hence $\left\|BD\right\| = \sqrt{\left(\frac{3}{2} - 2\sqrt{3}\right)^2 + \left(\frac{3\sqrt{3}}{2} + 2\right)^2} = \sqrt{25} = 5$
Specific behaviours
✓ Identifies the Cartesian coordinates of B and D ✓ Solves for $\left\|\overrightarrow{BD}\right\|$

Question 5 **(6 marks)**

(a) Solve the equation $X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ for the 2×2 matrix X. (4 marks)

Solution
$X\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ $X\left(\begin{bmatrix} 2 & -2 \\ -7 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ $X\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ $X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1}$ $X = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} \times \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ $X = \begin{bmatrix} 19 & 8 \\ -8 & -3 \end{bmatrix}$ **Or**   Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ Then $\begin{bmatrix} 3a-7b & -2a+5b \\ 3c-7d & -2c+5d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}$ Solve the two pairs of simultaneous equations to find $a=19,\ b=8,\ c=-8,\ d=-3$
Specific behaviours
✓ recognises that $X = X\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and simplifies the LS   ✓✓ post multiplies both sides of the equation by $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1}$, where $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ ✓ solves for X **Or**   ✓ substitutes $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and simplifies the LS   ✓✓ solves one pair of simultaneous equations   ✓ solves the second pair of simultaneous equations

(b) If A is a square matrix satisfying $A^2 - 2A + I = 0$, where I is the 2×2 identity matrix, determine an expression for A^{-1} in terms of A and I. (2 marks)

Solution
$A^2 - 2A + I = 0$ $I = 2A - A^2$ $A^{-1} = 2I - A$
Specific behaviours
✓ rearranges the equation to express I in terms of A ✓ multiplies both sides by A^{-1} and simplifies the RHS to establish the result

Or

Solution
$A^2 - 2A + I = 0$ $A - 2I + A^{-1} = 0$ multiply both sides by A^{-1} Hence $A^{-1} = 2I - A$
Specific behaviours
✓ multiplies both sides by A^{-1} ✓ rearranges the equation to express A^{-1} in terms of A and I

Question 6 **(5 marks)**

Evaluate exactly: $\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$

Solution
$\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$ Let $u = 2t^3+t+1$ Then $\dfrac{du}{dt} = 6t^2+1$ When $t=0,\ u=1;\quad t=10,\ u=2011$ Hence $\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \int_1^{2011} \frac{1}{\sqrt{u}}\,du$ Hence $\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \left[2\sqrt{u}\right]_1^{2011} = 2\left(\sqrt{2011}-1\right)$ Or $\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \int_{x=0}^{x=10} \frac{1}{\sqrt{u}}\,du$ Hence $\displaystyle\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = \left[2\sqrt{u}\right]_{t=0}^{t=10} = 2\left[\sqrt{2t^3+t+1}\right]_0^{10} = 2\left(\sqrt{2011}-1\right)$
Specific behaviours
✓ recognises the format $\displaystyle\int \frac{f'(x)}{f(x)}\,dx$ ✓ correctly rewrites the integral in terms of u ✓ correctly substitutes the new limits of integration (or replaces u with $2t^3+t+1$ after integration) ✓ integrates correctly ✓ solves exactly

Or

Solution
$\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt$ Let $u=\sqrt{(2t^3+t+1)}$ Then $\frac{du}{dt}=\frac{6t^2+1}{2\sqrt{(2t^3+t+1)}}$ When $t=0,\ u=1;\quad t=10,\ u=\sqrt{2011}$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = 2\int_1^{\sqrt{2011}} du$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = [2u]_1^{\sqrt{2011}} = 2\left(\sqrt{2011}-1\right)$ Or $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = 2\int_{x=0}^{x=10} du$ Hence $\int_0^{10} \frac{6t^2+1}{\sqrt{2t^3+t+1}}\,dt = [2u]_{t=0}^{t=10} = 2\left[\sqrt{2t^3+t+1}\right]_0^{10} = 2\left(\sqrt{2011}-1\right)$
Specific behaviours
✓ recognises the format $\int \frac{f'(x)}{f(x)}\,dx$ ✓ correctly rewrites the integral in terms of u ✓ correctly substitutes the new limits of integration (or replaces u with $\sqrt{2t^3+t+1}$ after integration) ✓ integrates correctly ✓ solves exactly

Question 7 **(9 marks)**

Consider the integrals $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ and $J = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx$.

(a) Use the substitution $u = a - x$ to show that $I = J$. (3 marks)

Solution
$I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ Let $u = a - x$ Then $du = -dx$; when $x = 0,\ u = a$; when $x = a,\ u = 0$ Then $I = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du)$ i.e. $I = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}\,du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx = J$
Specific behaviours
✓ correctly rewrites I new limits ✓ correctly rewrites integrand in terms of u ✓ recognises that $\int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}\,dx = J$

(b) By considering $I+J$, or otherwise, evaluate I in terms of a. (2 marks)

Solution
$I+J=\int_0^a \frac{f(x)}{f(x)+f(a-x)}dx+\int_0^a \frac{f(a-x)}{f(x)+f(a-x)}dx=\int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx$ Hence, since $I=J$ $\quad 2I=\int_0^a dx=a$ i.e. $\quad I=\frac{a}{2}$
Specific behaviours
✓ correctly simplifies to find $2I=\int_0^a dx$ ✓ integrates correctly

(c) Use the result from (b) to evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}\,dx$. (4 marks)

Solution
Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin\left(x+\frac{\pi}{4}\right)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\sin x}{\sin x + \cos x}\,dx$ i.e. $I = \sqrt{2}\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \sin\left(\frac{\pi}{2}-x\right)}\,dx$ (since $\sin\left(\frac{\pi}{2}-x\right) = \cos x$)   i.e. $I = \sqrt{2}\displaystyle\int_0^{a} \frac{f(x)}{f(x)+f(a-x)}\,dx$, where $f(x) = \sin x$ and $a = \frac{\pi}{2}$ Hence, using the result from (b), $I = \dfrac{\sqrt{2}\,\pi}{4}$
Specific behaviours
✓ correctly expands and simplifies $\sin\left(x+\frac{\pi}{4}\right)$ ✓ recognises that $\cos x = \sin\left(\frac{\pi}{2}-x\right)$ and hence that   ✓ the integral matches the pattern for $I = \displaystyle\int_0^{a} \frac{f(x)}{f(x)+f(a-x)}\,dx$ ✓ states the value of I.